7-42. Consider differential scattering by a magnetic obstacle (Fig. 7-17) and define

$$\kappa_e = j\omega(\epsilon - \epsilon_0) \qquad \kappa_m = j\omega(\mu - \mu_0)$$

Show that, instead of Eq. (7-143), we have

$$\text{Echo} = \frac{(\langle i,a\rangle/l)^2}{F(a,a) - \langle a,a\rangle}$$

where

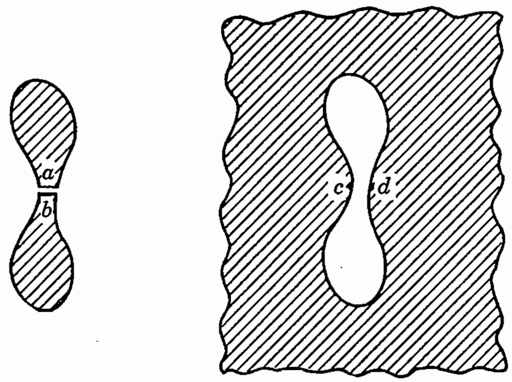
$$\langle i,a\rangle = \iiint (\mathbf{E}^i \cdot \mathbf{J}^a - \mathbf{H}^i \cdot \mathbf{M}^a)\, d\tau$$

$$F(a,a) = \iiint [\kappa_e^{-1}(J^a)^2 - \kappa_m^{-1}(M^a)^2]\, d\tau$$

$$\langle a,a\rangle = \iiint (\mathbf{E}^a \cdot \mathbf{J}^a - \mathbf{H}^a \cdot \mathbf{M}^a)\, d\tau$$

In the above formulas, $\mathbf{E}^i$, $\mathbf{H}^i$ is the incident field, $\mathbf{J}^a$ and $\mathbf{M}^a$ are the assumed electric and magnetic polarization currents on the obstacle, and $\mathbf{E}^a$, $\mathbf{H}^a$ is the field from $\mathbf{J}^a$, $\mathbf{M}^a$.

7-43. Figure 7-28*a* represents a metal antenna cut from a plane conductor and fed across the slot *ab*. Figure 7-28*b* represents the aperture formed by the remainder of the metal plane left after the metal antenna was cut. The aperture antenna, fed



(*a*) (*b*)

FIG. 7-28. (*a*) A sheet-metal antenna and (*b*) its complementary aperture antenna.

across *cd*, is said to be complementary to the metal antenna. Let Z_m be the input impedance of the metal antenna and Y_s be the input admittance to the slot antenna, and show that

$$\frac{Z_m}{Y_s} = \frac{\eta^2}{4}$$

Hint: Consider line integrals of **E** and **H** from *a* to *b* and *c* to *d*, and use duality.

7-44. Consider a narrow resonant slot of approximate length $\lambda/2$ in a conducting screen. Show that the transmission coefficient is

$$T \approx 0.52 \frac{\lambda}{w}$$

where w is the width of the slot. *Hint:* Use the result of Prob. 7-43 and assumptions similar to those of Prob. 7-39.

CHAPTER 8

MICROWAVE NETWORKS

8-1. Cylindrical Waveguides. Several special cases of the cylindrical waveguide, such as the rectangular and circular guides, already have been considered. We now wish to give a general treatment of cylindrical (cross section independent of z) waveguides consisting of a homogeneous isotropic dielectric bounded by a perfect electric conductor. Figure 8-1 represents the cross section of one such waveguide. Our formulation of the problem will be similar to that given by Marcuvitz.[1]

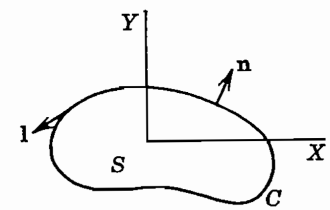


FIG. 8-1. Cross section of a cylindrical waveguide.

As shown in Sec. 3-12, general solutions for the field in a homogeneous region can be constructed from solutions to the Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0 \tag{8-1}$$

In cylindrical coordinates, this equation can be partially separated by taking

$$\psi = \Psi(x,y)Z(z) \tag{8-2}$$

The resultant pair of equations are

$$\nabla_t^2\Psi + k_c^2\Psi = 0 \tag{8-3}$$

$$\frac{d^2Z}{dz^2} + k_z^2 Z = 0 \tag{8-4}$$

where the separation constants k_z and k_c are related by

$$k_c^2 + k_z^2 = k^2 \tag{8-5}$$

and ∇_t is the two-dimensional (transverse to z) del operator

$$\nabla_t = \nabla - \mathbf{u}_z \frac{\partial}{\partial z} \tag{8-6}$$

[1] N. Marcuvitz, "Waveguide Handbook," MIT Radiation Laboratory Series, vol. 10, sec. 1-2, McGraw-Hill Book Company, Inc., New York, 1951.

Solutions to Eq. (8-4) are of the general form

$$Z(z) = Ae^{-jk_z z} + Be^{jk_z z} \tag{8-7}$$

which, for k_z real, is a superposition of $+z$ and $-z$ traveling waves. The k_z are determined from Eq. (8-5) after the k_c (cutoff wave numbers) are found by solving the boundary-value problem.

For TE modes, we take $\mathbf{F} = \mathbf{u}_z\psi^e$ (superscript e denotes TE) and determine

$$\mathbf{E}^e = -\mathbf{u}_x \frac{\partial \psi^e}{\partial y} + \mathbf{u}_y \frac{\partial \psi^e}{\partial x} = (\mathbf{u}_z \times \nabla_t \Psi^e) Z^e \tag{8-8}$$

The component of $\mathbf{E}$ tangential to the waveguide boundary C is

$$E_l^e = \mathbf{l} \cdot (\mathbf{u}_z \times \nabla_t \Psi^e) Z^e = (\mathbf{n} \cdot \nabla_t \Psi^e) Z^e$$

where $\mathbf{l}$ is the unit tangent to C and $\mathbf{n}$ is the unit normal to C (see Fig. 8-1). The boundary is perfectly conducting; hence $E_l = 0$ on C and

$$\frac{\partial \Psi^e}{\partial n} = 0 \qquad \text{on } C \tag{8-9}$$

The associated magnetic field is given by

$$\mathbf{H}^e = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}^e = \frac{1}{j\omega\mu}\left(\mathbf{u}_x \frac{\partial^2 \psi^e}{\partial x \partial z} + \mathbf{u}_y \frac{\partial^2 \psi^e}{\partial y \partial z} + \mathbf{u}_z k_c^2 \psi^e\right)$$

For more concise notation, we define a *transverse field vector* as

$$\mathbf{H}_t = \mathbf{H} - \mathbf{u}_z H_z \tag{8-10}$$

and rewrite the above as

$$\mathbf{H}_t^e = \frac{1}{j\omega\mu} (\nabla_t \Psi^e) \frac{dZ^e}{dz} \qquad H_z^e = \frac{k_c^2}{j\omega\mu} \Psi^e Z^e \tag{8-11}$$

It is evident from Eqs. (8-8) and (8-11) that lines of $\mathcal{E}$ and $\mathcal{H}_t$ are everywhere perpendicular to each other.

For TM modes, we take $\mathbf{A} = \mathbf{u}_z\psi^m$ (superscript m denotes TM) and, dual to Eq. (8-8), we determine

$$\mathbf{H}^m = -(\mathbf{u}_z \times \nabla_t \Psi^m) Z^m \tag{8-12}$$

Defining the transverse electric field vector $\mathbf{E}_t$ by Eq. (8-10) with $\mathbf{H}$ replaced by $\mathbf{E}$, we have, dual to Eq. (8-11),

$$\mathbf{E}_t^m = \frac{1}{j\omega\epsilon} (\nabla_t \Psi^m) \frac{dZ^m}{dz} \qquad E_z^m = \frac{k_c^2}{j\omega\epsilon} \Psi^m Z^m \tag{8-13}$$

From the second of these equations, it is evident that for E_z to vanish on C we must meet the boundary condition

$$\Psi^m = 0 \qquad \text{on } C \tag{8-14}$$

provided $k_c \neq 0$. Note that Eq. (8-14) also satisfies the condition $\mathbf{l} \cdot \mathbf{E}_t = 0$ on C. When the waveguide cross section is multiply connected, such as in coaxial lines, it is possible to have $k_c = 0$. In this case, the necessary boundary condition is $\Psi^m =$ constant on each conductor. The corresponding field is TEM to z and is a transmission-line mode.

It should be kept in mind that Eq. (8-3) subject to boundary conditions is an eigenvalue problem, giving rise to a discrete set of modes. These modes can be suitably ordered, and the various equations of this section then apply to each mode. It is convenient to introduce *mode functions* $\mathbf{e}(x,y)$ and $\mathbf{h}(x,y)$, *mode voltages* $V(z)$, and *mode currents* $I(z)$ according to

$$\begin{aligned} \mathbf{E}_t^e &= \mathbf{e}^e V^e \qquad & \mathbf{E}_t^m &= \mathbf{e}^m V^m \\ \mathbf{H}_t^e &= \mathbf{h}^e I^e \qquad & \mathbf{H}_t^m &= \mathbf{h}^m I^m \end{aligned} \tag{8-15}$$

Comparing Eqs. (8-15) with Eqs. (8-8) and (8-11), we see that we may choose

$$\begin{aligned} \mathbf{e}^e &= \mathbf{u}_z \times \nabla_t \Psi^e = \mathbf{h}^e \times \mathbf{u}_z \qquad & V^e &= Z^e \\ \mathbf{h}^e &= -\nabla_t \Psi^e = \mathbf{u}_z \times \mathbf{e}^e \qquad & I^e &= -\frac{1}{j\omega\mu}\frac{dZ^e}{dz} \end{aligned} \tag{8-16}$$

for TE modes, and, comparing Eqs. (8-15) with Eqs. (8-12) and (8-13),

$$\begin{aligned} \mathbf{e}^m &= -\nabla_t \Psi^m = \mathbf{h}^m \times \mathbf{u}_z \qquad & V^m &= -\frac{1}{j\omega\epsilon}\frac{dZ^m}{dz} \\ \mathbf{h}^m &= -\mathbf{u}_z \times \nabla_t \Psi^m = \mathbf{u}_z \times \mathbf{e}^m \qquad & I^m &= Z^m \end{aligned} \tag{8-17}$$

for TM modes. Furthermore, we *normalize* the mode vectors according to

$$\begin{aligned} \iint (e^e)^2\, ds &= \iint (h^e)^2\, ds = 1 \\ \iint (e^m)^2\, ds &= \iint (h^m)^2\, ds = 1 \end{aligned} \tag{8-18}$$

where the integration extends over the guide cross section. Hence, all amplitude factors are included in the V's and I's.

We shall now show that *all eigenvalues are real*. Consider the two-dimensional divergence theorem

$$\iint \nabla_t \cdot \mathbf{A}\, ds = \oint \mathbf{A} \cdot \mathbf{n}\, dl$$

and let $\mathbf{A} = \Psi^* \nabla_t \Psi$. Then,

$$\nabla_t \cdot \mathbf{A} = \nabla_t \Psi^* \cdot \nabla_t \Psi + \Psi^* \nabla_t^2 \Psi = |\nabla_t \Psi|^2 - k_c^2 |\Psi|^2$$

and the divergence theorem becomes

$$\iint (|\nabla_t \Psi|^2 - k_c^2 |\Psi|^2)\, ds = \oint \Psi^* \frac{\partial \Psi}{\partial n}\, dl$$

But the boundary conditions on the eigenfunction Ψ are either $\Psi = 0$ or $\partial\Psi/\partial n = 0$ on C. Hence, the right-hand term vanishes and

$$k_c^2 = \frac{\iint |\nabla_t \Psi|^2 \, ds}{\iint |\Psi|^2 \, ds} \tag{8-19}$$

The eigenvalue k_c^2 is therefore positive real. There is also no loss of generality if we take *all eigenfunctions* Ψ *to be real.* To justify this statement, suppose Ψ is not real, and let $\Psi = u + jv$. Then the Helmholtz equation is

$$\nabla_t^2\Psi + k_c^2\Psi = \nabla_t^2 u + k_c^2 u + j(\nabla_t^2 v + k_c^2 v) = 0$$

which, since k_c^2 is real, represents two Helmholtz equations for the real functions u and v. The boundary conditions, either

$$\Psi = u + jv = 0 \qquad \text{on } C$$

or

$$\frac{\partial \Psi}{\partial n} = \frac{\partial u}{\partial n} + j\frac{\partial v}{\partial n} = 0 \qquad \text{on } C$$

are satisfied independently by u and v; so u and v are solutions to the same boundary-value problem. Hence, u and v for a particular k_c can differ only by a constant, and Ψ is in phase over a guide cross section. We can take it to be real and include any phase in the V and I functions.

Let us now look at the propagation constant $\gamma = jk_z$. For ϵ and μ real, we have a cutoff wavelength

$$\lambda_c = \frac{2\pi}{k_c} \tag{8-20}$$

and a cutoff frequency

$$f_c = \frac{k_c}{2\pi\sqrt{\epsilon\mu}} \tag{8-21}$$

Then, from Eq. (8-5), we have the propagation constant given by

$$\gamma = jk_z = \begin{cases} j\beta = jk\sqrt{1 - \left(\dfrac{f_c}{f}\right)^2} & f > f_c \\ \alpha = k_c\sqrt{1 - \left(\dfrac{f}{f_c}\right)^2} & f < f_c \end{cases} \tag{8-22}$$

These are, of course, just the relationships that we previously established for the rectangular and circular waveguides. Figure 2-18 illustrates the behavior of α and β versus f. When the mode is propagating ($f > f_c$), the concepts of *guide wavelength*,

$$\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda}{\sqrt{1 - (f_c/f)^2}} \tag{8-23}$$

where λ is the intrinsic wavelength in the dielectric, and *guide phase velocity*,

$$v_g = \frac{\omega}{\beta} = \frac{v_p}{\sqrt{1 - (f_c/f)^2}} \tag{8-24}$$

where v_p is the intrinsic phase velocity, are useful. These parameters are discussed in Sec. 2-7.

Turning now to the mode voltages and currents, we see from their definitions [Eqs. (8-16) and (8-17)] that V and I satisfy Eq. (8-4). Hence, in general they are of the form of Eq. (8-7), or

$$\begin{aligned} V(z) &= V^+e^{-\gamma z} + V^-e^{\gamma z} \\ I(z) &= I^+e^{-\gamma z} + I^-e^{\gamma z} \end{aligned} \tag{8-25}$$

where superscripts + and − denote positively and negatively traveling (or attenuating) wave components. Also, from Eqs. (8-4), (8-16), and (8-17) it is apparent that

$$\frac{V^+}{I^+} = Z_0 \qquad \frac{V^-}{I^-} = -Z_0 \tag{8-26}$$

where the *characteristic impedance* Z_0 is, for TE modes,

$$Z_0^e = \frac{j\omega\mu}{\gamma} = \begin{cases} \dfrac{\omega\mu}{\beta} = \dfrac{\eta}{\sqrt{1 - (f_c/f)^2}} & f > f_c \\ \dfrac{j\omega\mu}{\alpha} = \dfrac{j\omega\mu}{k_c\sqrt{1 - (f/f_c)^2}} & f < f_c \end{cases} \tag{8-27}$$

and, for TM modes,

$$Z_0^m = \frac{\gamma}{j\omega\epsilon} = \begin{cases} \dfrac{\beta}{\omega\epsilon} = \eta\sqrt{1 - \left(\dfrac{f_c}{f}\right)^2} & f > f_c \\ \dfrac{\alpha}{j\omega\epsilon} = \dfrac{k_c}{j\omega\epsilon}\sqrt{1 - \left(\dfrac{f}{f_c}\right)^2} & f < f_c \end{cases} \tag{8-28}$$

Note that these are just the characteristic wave impedances that we previously defined for rectangular and circular waveguides. Figure 4-3 illustrates the behavior of the Z_0's versus frequency. Finally, from Eqs. (8-4), (8-16), and (8-17), we can show that V and I also satisfy the *transmission-line equations*

$$\begin{aligned} \frac{dV}{dz} &= -\gamma Z_0 I \\ \frac{dI}{dz} &= -\gamma Y_0 V \end{aligned} \tag{8-29}$$

where $Y_0 = 1/Z_0$ is the *characteristic admittance.* Hence, the analogy

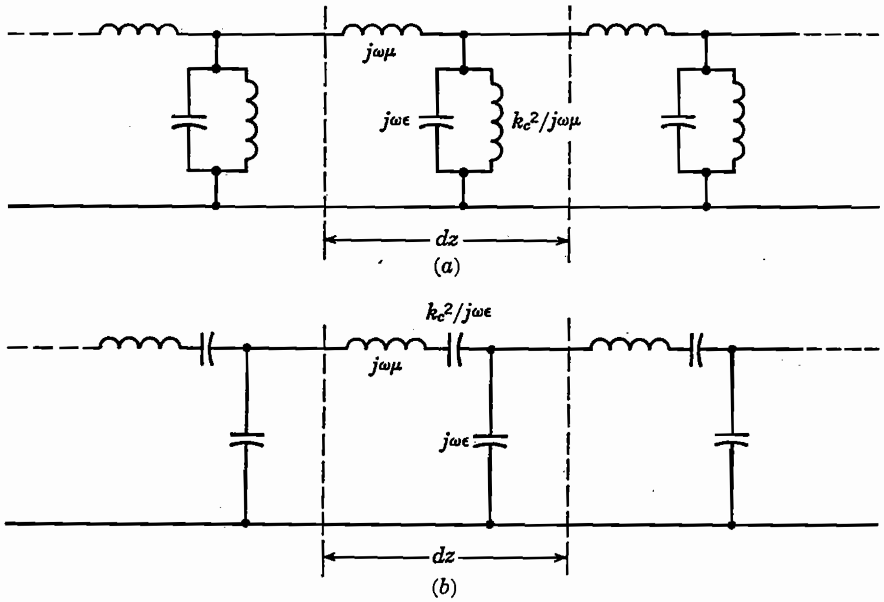


FIG. 8-2. Equivalent transmission lines for waveguide modes (series elements labeled in ohms, shunt elements in mhos). (*a*) TE modes, (*b*) TM modes.

with transmission lines is complete, and all of the techniques for analyzing transmission lines can be applied to each waveguide mode.[1]

We may define an *equivalent transmission line* for each waveguide mode as one for which γ and Z_0 are the same as those of the waveguide mode. Such an equivalent circuit may help us to visualize waveguide behavior by presenting it in terms of the more familiar transmission-line behavior. For a dissipationless transmission line, we have

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{X}{B}}$$
$$\gamma = \sqrt{ZY} = j\sqrt{XB}$$

(see Sec. 2-6). Equating the above Z_0 and γ to those of a TE waveguide mode, we obtain

$$jX = j\omega\mu \qquad jB = j\omega\epsilon + \frac{k_c^2}{j\omega\mu} \tag{8-30}$$

Thus, the transmission line equivalent to a TE mode is as shown in Fig. 8-2*a*. Similarly, for a TM mode we obtain

$$jX = j\omega\mu + \frac{k_c^2}{j\omega\epsilon} \qquad jB = j\omega\epsilon \tag{8-31}$$

[1] For example, see Wilbur LePage and Samuel Seely, "General Network Analysis," Chaps. 9 and 10, McGraw-Hill Book Company, Inc., New York, 1952.

The transmission line equivalent to a TM mode is therefore as shown in Fig. 8-2*b*. If the dielectric is lossy, the equivalent transmission will also have resistances, obtained by replacing $j\omega\epsilon$ by $\sigma + j\omega\epsilon$ in Eqs. (8-30) and (8-31). In the light of filter theory, we can recognize the equivalent transmission lines as high-pass filters.

The power transmitted along the waveguide is, of course, obtained by integrating the Poynting vector over the guide cross section. Hence, for the $+z$ direction,

$$P_z = \iint \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{u}_z \, ds = VI^* \iint \mathbf{e} \times \mathbf{h}^* \cdot \mathbf{u}_z \, ds$$
$$= VI^* \iint e^2 \, ds = VI^* \tag{8-32}$$

and the time-average power transmitted is

$$\bar{\mathcal{P}}_z = \operatorname{Re}(VI^*) \tag{8-33}$$

Hence, in terms of the mode voltage and current, power is calculated by the usual circuit-theory formulas.

It is also worthwhile to note that the mode patterns, that is, pictures of lines of $\mathcal{E}$ and $\mathcal{H}$ at some instant, can be obtained directly from the Ψ's. For TE modes, $\mathbf{H}_t$ is proportional to $\nabla_t\Psi^e$, and $\mathbf{E}$ is perpendicular to $\mathbf{H}_t$. Hence, *lines of constant Ψ^e are also lines of instantaneous $\mathcal{E}$.* Lines of instantaneous $\mathcal{H}_t$ are everywhere perpendicular to lines of instantaneous $\mathcal{E}$. Similarly, for TM modes, *lines of constant Ψ^m are also lines of instantaneous $\mathcal{H}$*, and lines of instantaneous $\mathcal{E}_t$ are everywhere perpendicular to lines of instantaneous $\mathcal{H}$. It is therefore quite easy to sketch the mode patterns directly from the eigenfunctions Ψ.

Recognizing that the general exposition of cylindrical waveguides has been quite lengthy, let us summarize the results. Table 8-1 lists the more important relationships that we have derived. Those equations common to both TE and TM modes are written centered in the table. Keep in mind that all of the equations apply to *each mode* and that many modes may exist simultaneously in any given waveguide.

Finally, for future reference, let us tabulate the normalized eigenfunctions for the special cases already treated. For the rectangular waveguide of Fig. 2-16, we can pick the Ψ's from Eqs. (4-19) and (4-21) and normalize them according to Eq. (8-18). The result is

$$\Psi_{mn}{}^e = \frac{1}{\pi}\sqrt{\frac{ab\epsilon_m\epsilon_n}{(mb)^2 + (na)^2}} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$
$$\Psi_{mn}{}^m = \frac{2}{\pi}\sqrt{\frac{ab}{(mb)^2 + (na)^2}} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{8-34}$$

where $m, n = 0, 1, 2, \ldots$, ($m = n = 0$ excepted). Similarly, for the

TABLE 8-1. SUMMARY OF EQUATIONS FOR THE CYLINDRICAL WAVEGUIDE (TEM MODES NOT INCLUDED)

	TE modes	TM modes
Transverse Helmholtz equation	$\nabla_t^2\Psi + k_c^2\Psi = 0$	
Boundary relations	$\frac{\partial \Psi^e}{\partial n} = 0$ on C	$\Psi^m = 0$ on C
Mode vectors	$\mathbf{e}^e = \mathbf{u}_z \times \nabla_t\Psi^e$ $\mathbf{h}^e = -\nabla_t\Psi^e$	$\mathbf{e}^m = -\nabla_t\Psi^m$ $\mathbf{h}^m = -\mathbf{u}_z \times \nabla_t\Psi^m$
	$\mathbf{e} = \mathbf{h} \times \mathbf{u}_z$ $\mathbf{h} = \mathbf{u}_z \times \mathbf{e}$	
Normalization	$\iint e^2\, ds = \iint h^2\, ds = 1$	
Propagation constant	$\gamma = jk_z = \begin{cases} j\beta = jk\sqrt{1-(f_c/f)^2} & f > f_c \\ \alpha = k_c\sqrt{1-(f/f_c)^2} & f < f_c \end{cases}$	
Characteristic Z and Y	$Z_0^e = \frac{j\omega\mu}{\gamma} = \frac{1}{Y_0^e}$	$Z_0^m = \frac{\gamma}{j\omega\epsilon} = \frac{1}{Y_0^m}$
Transmission-line equations	$\frac{dV}{dz} + \gamma Z_0 I = 0$ $\frac{dI}{dz} + \gamma Y_0 V = 0$	
Mode voltage and current	$V = V^+e^{-\gamma z} + V^-e^{\gamma z}$ $I = \frac{1}{Z_0}(V^+e^{-\gamma z} - V^-e^{\gamma z})$	
Transverse field	$\mathbf{E}_t = \mathbf{e}V$ $\mathbf{H}_t = \mathbf{h}I$	
Longitudinal field	$H_z^e = \frac{k_c^2}{j\omega\mu}\Psi^e V^e$	$E_z^m = \frac{k_c^2}{j\omega\epsilon}\Psi^m I^m$
z-directed power	$P_z = VI^*$	

circular waveguide of Fig. 5-2, we can pick the Ψ's from Eqs. (5-23) and (5-27) and normalize them. The result is

$$\Psi_{np}^e = \sqrt{\frac{\epsilon_n}{\pi[(x'_{np})^2 - n^2]}}\,\frac{J_n(x'_{np}\rho/a)}{J_n(x'_{np})}\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix}$$
$$\Psi_{np}^m = \sqrt{\frac{\epsilon_n}{\pi}}\,\frac{J_n(x_{np}\rho/a)}{x_{np}\,J_{n+1}(x_{np})}\begin{Bmatrix}\sin n\phi\\ \cos n\phi\end{Bmatrix} \tag{8-35}$$

where $n = 0, 1, 2, \ldots$, and $p = 1, 2, 3, \ldots$. The x_{np} are given by Table 5-2, and the x'_{np} are given by Table 5-3. Normalized eigenfunctions for the parallel-plate guide are given in Prob. 8-1. Normalized eigenfunctions for the coaxial and elliptic waveguides are given by Marcuvitz.[1]

8-2. Modal Expansions in Waveguides. An arbitrary field inside a section of waveguide can be expanded as a sum over all possible modes. This concept was used in Sec. 4-4 for the special case of the rectangular waveguide. We now wish to consider such expansions for cylindrical waveguides in general. The equations in Sec. 8-1 apply to each mode. Henceforth, to identify a particular mode, we shall use the subscript i to denote the mode number.

Let us first show that each mode vector $\mathbf{e}_i$ is orthogonal to all other mode vectors. For this, we shall use the divergence theorem in two dimensions,

$$\iint \nabla_t \cdot \mathbf{A}\, ds = \oint \mathbf{A}\cdot\mathbf{n}\, dl$$

Green's first identity in two dimensions,

$$\iint (\nabla_t\psi \cdot \nabla_t\phi + \psi\nabla_t^2\phi)\, ds = \oint \psi \frac{\partial\phi}{\partial n}\, dl$$

and Green's second identity in two dimensions,

$$\iint (\psi\nabla_t^2\phi - \phi\nabla_t^2\psi)\, ds = \oint \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dl$$

First, consider two TE modes and form the product

$$\mathbf{e}_i^e \cdot \mathbf{e}_j^e = \mathbf{h}_i^e \cdot \mathbf{h}_j^e = \nabla_t\Psi_i^e \cdot \nabla_t\Psi_j^e$$

Letting $\psi = \Psi_i^e$ and $\phi = \Psi_j^e$ in Green's first identity, we obtain

$$\iint \mathbf{e}_i^e \cdot \mathbf{e}_j^e\, ds = -(k_{cj}^e)^2 \iint \Psi_i^e\Psi_j^e\, ds$$

Using the same substitution in Green's second identity, we have

$$[(k_{ci}^e)^2 - (k_{cj}^e)^2]\iint \Psi_i^e\Psi_j^e\, ds = 0$$

[1] N. Marcuvitz, "Waveguide Handbook," MIT Radiation Laboratory Series, vol. 10, chap. 2, McGraw-Hill Book Company, Inc., New York, 1951.

Hence, if $k_{ci}^e \neq k_{cj}^e$, the integral must vanish, and the preceding equation becomes[1]

$$\iint \mathbf{e}_i^e \cdot \mathbf{e}_j^e \, ds = 0 \qquad i \neq j \tag{8-36}$$

A dual analysis applies to the TM modes, and we have

$$\iint \mathbf{e}_i^m \cdot \mathbf{e}_j^m \, ds = 0 \qquad i \neq j \tag{8-37}$$

Finally, we must consider the TE-TM cross products

$$\mathbf{e}_i^e \cdot \mathbf{e}_j^m = \mathbf{h}_i^e \cdot \mathbf{h}_j^m = -(\mathbf{u}_z \times \nabla_t \Psi_i^e) \cdot \nabla_t \Psi_j^m$$

If we let $\mathbf{A} = \Psi_j^m \mathbf{u}_z \times \nabla_t \Psi_i^e$ in the divergence theorem, the contour integral vanishes because of the boundary conditions, and we obtain

$$\iint \nabla_t \Psi_j^m \cdot \mathbf{u}_z \times \nabla_t \Psi_i^e \, ds = 0$$

Comparing the preceding two equations, we see that

$$\iint \mathbf{e}_i^e \cdot \mathbf{e}_j^m \, ds = 0 \qquad \text{for all } i, j \tag{8-38}$$

The orthogonality relationships [Eqs. (8-36) to (8-38)] also are valid for the e's replaced by the **h**'s.

At any cross section along a cylindrical waveguide, the field can be expressed as a summation over all possible modes:

$$\begin{aligned} \mathbf{E}_t &= \sum_i \mathbf{e}_i^e V_i^e + \mathbf{e}_i^m V_i^m \\ \mathbf{H}_t &= \sum_i \mathbf{h}_i^e I_i^e + \mathbf{h}_i^m I_i^m \end{aligned} \tag{8-39}$$

Because of the orthogonality of the mode vectors, we can determine the mode voltages and/or mode currents at any cross section by multiplying each side of Eqs. (8-39) by an arbitrary mode vector and integrating over the guide cross section. Noting that the mode vectors are normalized, we obtain

$$\begin{aligned} \iint \mathbf{E}_t \cdot \mathbf{e}_i^p \, ds &= V_i^p \\ \iint \mathbf{H}_t \cdot \mathbf{h}_i^p \, ds &= I_i^p \end{aligned} \tag{8-40}$$

where $p = e$ or m. Since there are two independent constants in V and I for each mode, as shown by Eqs. (8-25) and (8-26), we need two "cross-sectional" boundary conditions. These may be (1) matched waveguide and $\mathbf{E}_t$ over one cross section, (2) matched waveguide and $\mathbf{H}_t$ over one cross section, (3) $\mathbf{E}_t$ over two cross sections, (4) $\mathbf{H}_t$ over two cross sections, and (5) $\mathbf{E}_t$ over one cross section and $\mathbf{H}_t$ over another cross section. The solutions of Sec. 4-9 are examples of case (1). Furthermore, when we have currents in a waveguide, we can obtain additional cases involving discontinuities in $\mathbf{E}_t$ and/or $\mathbf{H}_t$ over waveguide cross sections. The solutions of Sec. 4-10 are examples of this situation.

[1] A discrete spectrum of eigenvalues is assumed. However, orthogonal sets of mode functions for degenerate cases can also be found.

It is also of interest to note that, when many modes exist simultaneously in a cylindrical waveguide, *each mode propagates energy as if it exists alone.* Hence, the equivalent circuit of a section of waveguide in which N modes exist is N separate transmission lines of the form of Fig. 8-2. To show this power orthogonality, we calculate the z-directed complex power

$$\begin{aligned} P_z &= \iint \mathbf{E} \times \mathbf{H}^* \cdot \mathbf{u}_z \, ds = \iint \Big(\sum_i \mathbf{e}_i V_i\Big) \times \Big(\sum_j \mathbf{h}_j I_j^*\Big) \cdot \mathbf{u}_z \, ds \\ &= \sum_{i,j} V_i I_j^* \iint \mathbf{e}_i \cdot \mathbf{e}_j \, ds = \sum_i V_i I_i^* \end{aligned} \tag{8-41}$$

which is a summation of the powers carried by each mode. (We have used the indices i and j to order both TE and TM modes in the above proof.) The energy stored per unit length in a waveguide is also the sum of the energies stored in each mode (see Prob. 8-3).

8-3. The Network Concept. In Sec. 3-8, we saw that, given N sets of "circuit" terminals, the voltages at the terminals were related to the currents by an impedance matrix. This impedance matrix was shown to be symmetrical, that is, the usual circuit-theory reciprocity applied if the medium was isotropic. We shall now show that the same network formulation applies if, instead of circuit voltages and currents, the modal voltages and currents of waveguide "ports" are used.

Let Fig. 8-3 represent a general "microwave network," that is, a system for which a closed surface separating the network from the rest of space can be found such that $\mathbf{n} \times \mathbf{E} = 0$ on the surface except over one or more waveguide cross sections. Suppose that only one mode propagates

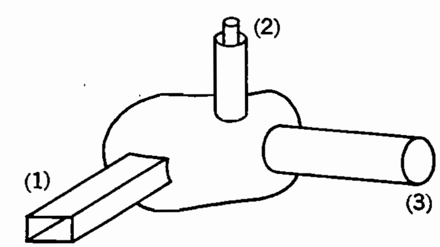


FIG. 8-3. A microwave network.